Sapt. 27 (10-14 mai)

Functia de gradul al II-lea

Functia $f : R \longrightarrow R$, $f(x) = ax^2 + bx + c$, $a,b,c \in R$, $a \neq 0$ se numeste functie de gradul al II-lea

Expresia legii de corespondenta a functiei de gradul al II-lea se poate scrie si astfel:

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left[\left(x^2 + 2\cdot\frac{b}{2a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2} + \frac{c}{a}\right] =$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right] = a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}$$, unde $\Delta = b^2 - 4ac$.

$$\boxed{f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}, \quad \Delta = b^2 - 4ac}$$ se numeste forma canonica a functiei de gradul al II-lea

Folosind forma canonica se deduc urmatoarele:

1) $a > 0 \Rightarrow y_{min} = \frac{-\Delta}{4a}$ (valoarea minima a functiei) , $x_{min} = \frac{-b}{2a}$

2) $a < 0 \Rightarrow y_{max} = \frac{-\Delta}{4a}$ (valoarea maxima a functiei) , $x_{max} = \frac{-b}{2a}$

Punctul $V\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$ se numeste punctul de extrem al graficului functiei de gr. II sau varful graficului.

Aplicatie

Sa se determine functia f de gradul al doilea daca f(1)=0, f(0)=1, f(-1)=8.

Rezolvare: $f(x) = ax^2 + bx + c$, $a, b, c \in R$, $a \neq 0$

$$\begin{cases} f(1) = 0 \\ f(0) = 1 \\ f(-1) = 8 \end{cases} \Leftrightarrow \begin{cases} a\cdot 1^2 + b\cdot 1 + c = 0 \\ a\cdot 0^2 + b\cdot 0 + c = 1 \\ a\cdot(-1)^2 + b(-1) + c = 8 \end{cases} \Leftrightarrow \begin{cases} a + b + c = 0 \\ \boxed{c = 1} \\ a - b + c = 8 \end{cases} \Rightarrow \begin{cases} a + b + 1 = 0 \\ a - b + 1 = 8 \end{cases} \Leftrightarrow \begin{cases} a + \cancel{b} = -1 \\ a - \cancel{b} = 7 \end{cases}$$

$$2a = 6$$

$\Rightarrow \boxed{a = 3}$

$a + b = -1 \Leftrightarrow 3 + b = -1 \Leftrightarrow \boxed{b = -4}$

Deci functia cautata este $f(x) = 3x^2 - 4x + 1$.

Aplicatie

Sa se scrie sub forma canonica functia $f(x) = 2x^2 - 4x + 5$, $x \in \mathbb{R}$.

Rezolvare

$$f(x) = \underbrace{ax^2 + bx + c}_{\text{forma generala}} = \underbrace{a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}}_{\text{forma canonica}}$$

$$f(x) = 2x^2 - 4x + 5$$

$$a = 2 \quad b = -4 \quad c = 5$$

$$\Delta = b^2 - 4ac = (-4)^2 - 4 \cdot 2 \cdot 5 = 16 - 40 = -24 < 0$$

$$f(x) = 2x^2 - 4x + 5 = 2 \cdot \left(x - \frac{4}{4}\right)^2 + \frac{24}{8} = 2(x-1)^2 + 3$$

forma canonica cautata

Aplicatie

Sa se rezolve ecuatia de gradul al II-lea cunoscand o solutie: $3x^2 + (m^2 - 3m)x - 4 = 0$, $x_1 = -2$.

Rezolvare:

$$x_1 = -2 \Rightarrow 3(-2)^2 + (m^2 - 3m) \cdot (-2) - 4 = 0 \Rightarrow$$

$$\Rightarrow 3 \cdot 4 - 2m^2 + 6m - 4 = 0 \Leftrightarrow 2m^2 - 6m - 8 = 0 \Leftrightarrow m^2 - 3m - 4 = 0$$

$$m_{1,2} = \frac{3 \pm \sqrt{9+16}}{2} = \begin{cases} \frac{3+5}{2} = 4 \\ \frac{3-5}{2} = \frac{-2}{2} = -1 \end{cases}$$

Daca $m = 4$ atunci avem ecuatia $3x^2 + (4^2 - 3 \cdot 4) \cdot x - 4 = 0 \Leftrightarrow 3x^2 + 4x - 4 = 0$

$$\Delta = 16 - 4 \cdot 3 \cdot (-4) = 16 + 48 = 64, \quad x_{1,2} = \frac{-4 \pm 8}{6} = \begin{cases} 4/6 = 2/3 \\ -12/6 = \boxed{-2} \end{cases}$$

Daca $m = -1$ atunci avem ecuatia $3x^2 + (1+3)x - 4 = 0 \Leftrightarrow$
$3x^2 + 4x - 4 = 0$ aceeasi ecuatie ca pentru $m = 4$

Deci solutiile ecuatie in enunt sunt $x = 2/3$ si $x = -2$

Reprezentarea grafica a functiei de gradul al II -lea

Dreapta $x = -\frac{b}{2a}$ este axa de simetrie pentru graficul functiei de gradul al II-lea.

Intersectia graficului cu axa Oy se determina calculand f(0).
Intersectia graficului cu axa Ox se determina rezolvand f(x) = 0.
Varful graficului este $V\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$

Ex: $f(x) = x^2 - 3x + 2$

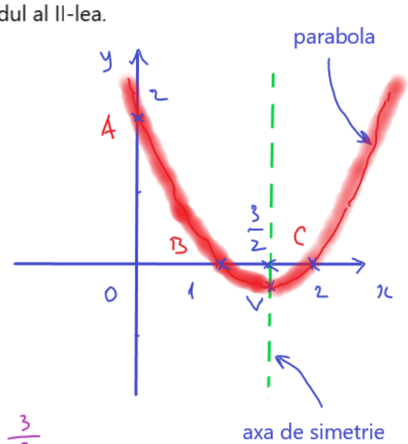
x	$-\infty$	0	1	$3/2$	2	$+\infty$
$f(x)$		2 A	0 B	$-1/4$	0 C	

$$f(0) = 0^2 - 3 \cdot 0 + 2 = 2$$

$$f(x) = 0 \Leftrightarrow x^2 - 3x + 2 = 0, \quad \Delta = (-3)^2 - 4 \cdot 2 = 9 - 8 = 1$$

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{3+1}{2} = \frac{4}{2} = 2 \qquad x_{min} = \frac{-b}{2a} = \frac{3}{2}$$

$$x_2 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{3-1}{2} = \frac{2}{2} = 1 \qquad y_{min} = \frac{-\Delta}{4a} = \frac{-1}{4}$$



Aplicatie

Sa se reprezinte grafic functia de gradul II $f(x) = 3x^2 - x - 10$, $x \in \mathbb{R}$.

Rezolvare:

x	$-\infty$		$-5/3$		0		$1/6$		2		$+\infty$
$f(x)$		$\searrow$	0	$\searrow$	-10	$\searrow$	$-\frac{121}{12}$	$\nearrow$	0	$\nearrow$	

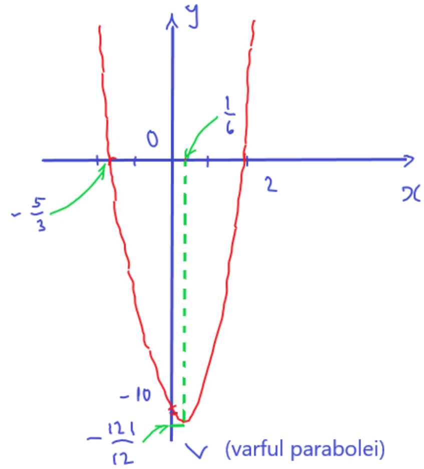
$f(x) = 0 \Leftrightarrow 3x^2 - x - 10 = 0$

$\Delta = b^2 - 4ac = (-1)^2 - 4 \cdot 3 \cdot (-10) = 121$

$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{1 \pm 11}{6}$; $\frac{12}{6} = 2$; $\frac{-10}{6} = -\frac{5}{3}$

$x_V = -\frac{b}{2a} = \frac{1}{6}$

$y_V = -\frac{\Delta}{4a} = \frac{-121}{12}$ $\qquad f(0) = -10$



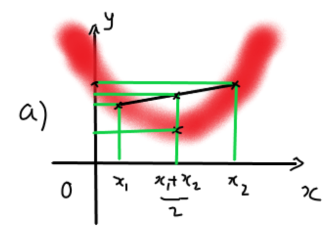
Proprietati ale punctelor graficului functiei de gradul al II-lea

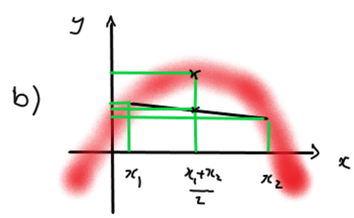
1) Oricare trei puncte distincte ale graficului functiei de gradul al II-lea sunt necoliniare.

2) Fie $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax^2 + bx + c$, $a \neq 0$

a) Daca $a > 0$ atunci $\frac{f(x_1) + f(x_2)}{2} \geq f\left(\frac{x_1 + x_2}{2}\right)$, $\forall x_1, x_2 \in \mathbb{R}$ (functie convexa)

b) Daca $a < 0$ atunci $\frac{f(x_1) + f(x_2)}{2} \leq f\left(\frac{x_1 + x_2}{2}\right)$, $\forall x_1, x_2 \in \mathbb{R}$ (functie concava)





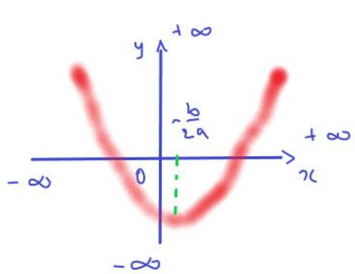
Interpretarea geometrica a proprietatilor algebrice ale functiei de gradul al II-lea

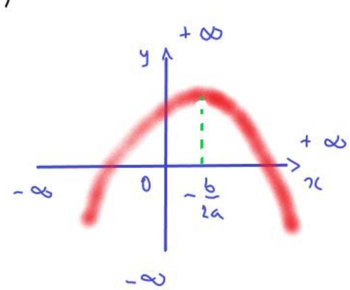
Monotonia

Fie $f(x) = ax^2 + bx + c$, $a,b,c \in \mathbb{R}$, $a \neq 0$

Daca $a > 0$ atunci f strict descrescatoare pe intervalul $\left(-\infty, -\frac{b}{2a}\right)$ si strict crescatoare pe intervalul $\left[-\frac{b}{2a}, +\infty\right)$

Daca $a < 0$ atunci f strict crescatoare pe intervalul $\left(-\infty, -\frac{b}{2a}\right)$ si strict descrescatoare pe intervalul $\left[-\frac{b}{2a}, +\infty\right)$





Relatiile lui Viete

$ax^2 + bx + c = 0,\ a \neq 0,\ a,b,c \in R$ forma generala a ecuatiei de gradul al II-lea

$a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a} = 0,\ a \neq 0,\ a,b,c \in R$ forma canonica a ecuatiei de gradul al II-lea

$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$, $\Delta = b^2 - 4ac$ formula de calcul pentru solutiile ecuatiei de gradul al II-lea

Daca $\Delta > 0$ atunci $x_1, x_2 \in R$ si $x_1 \neq x_2$
Daca $\Delta = 0$ atunci $x_1, x_2 \in R$ si $x_1 = x_2$
Daca $\Delta < 0$ atunci $x_1, x_2 \notin R$

Cand se cunosc radacinile ecuatiei de gradul al II-lea aceasta poate fi scrisa si sub forma

$a(x - x_1)(x - x_2) = 0$

Relatii intre radacinile si coeficientii ecuatiei de gradul al II-lea

$$\begin{cases} S = x_1 + x_2 = -\frac{b}{a} \\ P = x_1 \cdot x_2 = \frac{c}{a} \end{cases}$$ (relatiile lui Viete)

Astfel ecuatia de gradul al II-lea se mai poate scrie

$x^2 - Sx + P = 0$

Aplicatii

Sa se rezolve sistemul de ecuatii $\begin{cases} x + y = 3 \\ x \cdot y = -10 \end{cases}$

Rezolvare:

Se formeaza ecuatia de gradul al II-lea de radacini x si y : $t^2 - St + P = 0$

$t^2 - 3t - 10 = 0$, $\Delta = b^2 - 4ac = (-3)^2 - 4 \cdot (-10) = 9 + 40 = 49$

$t_1 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{3+7}{2} = \frac{10}{2} = 5$

$t_2 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{3-7}{2} = \frac{-4}{2} = -2$

t_1 si t_2 sunt radacinile ecuatiei, adica x sau y.
Prin urmare

$\begin{cases} x = 5 \\ y = -2 \end{cases}$ sau $\begin{cases} x = -2 \\ y = 5 \end{cases}$

Aplicatie

Sa se rezolve sistemul de ecuatii $\begin{cases} x^2 + y^2 = 2 \\ xy = -1 \end{cases}$ (sistem simetric)

Rezolvare:

$\begin{cases} x^2 + y^2 = 2 \\ xy = -1 \end{cases} \Leftrightarrow \begin{cases} (x+y)^2 - 2xy = 2 \\ xy = -1 \end{cases} \xrightarrow[x \cdot y = P]{x + y = S} \begin{cases} S^2 - 2P = 2 \\ P = -1 \end{cases} \Leftrightarrow \begin{cases} S^2 + 2 = 2 \\ P = -1 \end{cases}$

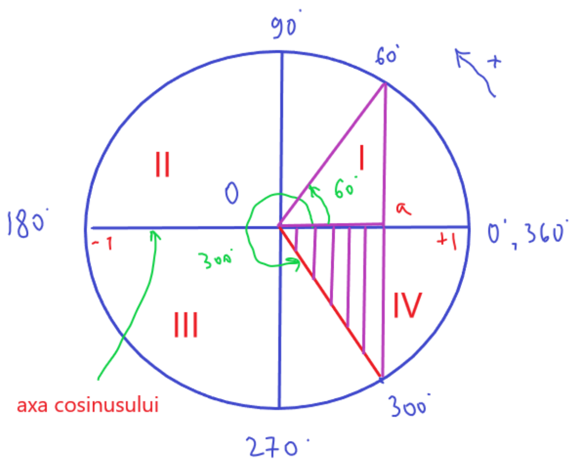
$\Leftrightarrow \begin{cases} S^2 = 0 \\ P = -1 \end{cases} \Leftrightarrow \begin{cases} S = 0 \\ P = -1 \end{cases}$

$t^2 - St + P = 0 \Leftrightarrow t^2 - 0 \cdot t + (-1) = 0 \Leftrightarrow t^2 - 1 = 0 \Leftrightarrow t_1 = 1,\ t_2 = -1$

$\Rightarrow \begin{cases} x = 1 \\ y = -1 \end{cases}$ sau $\begin{cases} y = 1 \\ x = -1 \end{cases}$

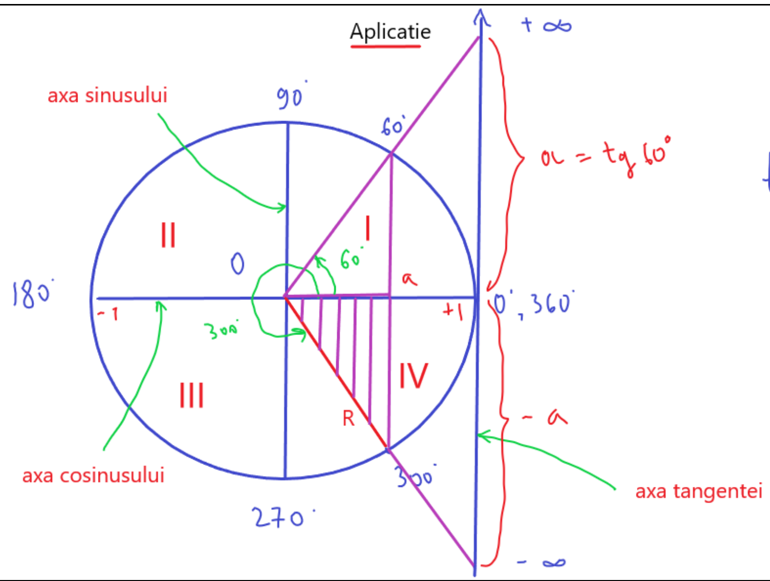
$\boxed{(a+b)^2 = a^2 + b^2 + 2ab \Rightarrow a^2 + b^2 = (a+b)^2 - 2ab}$

Reducerea la primul cadran. Aplicatii



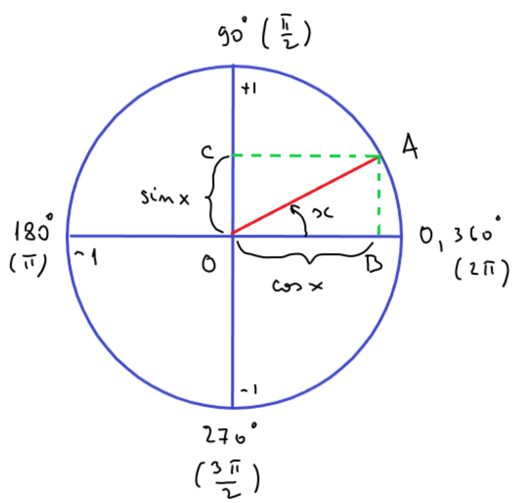
$$\cos 300^\circ = \cos 60^\circ = \frac{1}{2} = a$$

Aplicatie



$$tg\,300^\circ = -tg\,60^\circ = -\sqrt{3}$$

Formula fundamentala a trigonometriei



$\triangle AOB$ dreptunghic

$AO^2 = AB^2 + BO^2$ (teorema lui Pitagora)

$AB = CO = \sin x$

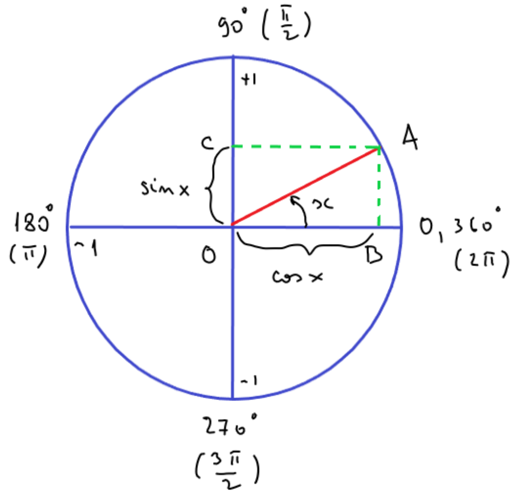
$BO = \cos x$

Inlocuind in relatia de mai sus obtinem

$$(\sin x)^2 + (\cos x)^2 = 1$$

formula fundamentala a trigonometriei

Formule trigonometrice



$$\cos(90° - x) = \sin x$$
$$\sin(90° - x) = \cos x$$
$$tg(90° - x) = ctg\, x$$
$$ctg(90° - x) = tg\, x$$

$$\cos(a-b) = \cos a \cdot \cos b + \sin a \cdot \sin b$$
$$\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$$
$$\sin(a+b) = \sin a \cdot \cos b + \cos a \cdot \sin b$$
$$\sin(a-b) = \sin a \cdot \cos b - \cos a \cdot \sin b$$

Aplicatie

$$\sin(a+b) = \sin a \cdot \cos b + \cos a \cdot \sin b$$

1) $\sin 105° = \sin(45° + 60°) = \sin 45° \cdot \cos 60° + \cos 45° \cdot \sin 60° = \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$

2) $\sin 15° = \sin(60° - 45°) = \sin 60° \cdot \cos 45° - \cos 60° \cdot \sin 45° =$
$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$

3) $\cos 75° = \cos(30° + 45°) = \cos 30° \cdot \cos 45° - \sin 30° \cdot \sin 45° =$
$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4} = \sin(90° - 75°) =$
$= \sin 15°$

	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tg	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
ctg	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

$$\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$$